Advanced Topics in Physics ♦ Vector Problems

1. Write each of the following vectors in rectangular form, i.e. $\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j}$:
   a) $2 \measuredangle 60°$.
   b) $\mathbf{C} = \mathbf{A} + \mathbf{B}$ and $\mathbf{D} = \mathbf{A} - \mathbf{B}$, where $\mathbf{A} = 12\,\mathbf{i} + 7\,\mathbf{j}$ and $\mathbf{B} = 6\,\mathbf{i} - 2\,\mathbf{j}$.
   c) The vector sum of 5 @ 70° + 3 @ 250°
   d) A unit vector @ 30°.

2. Find the magnitude and direction (polar form) of 1 (c).

3. By inspection, write the position vector $\mathbf{r}(t)$ for a particle with each of the following motions, and check your answer using calculus:
   a) Stationary at (3,2).
   b) Moving in the $+y$ direction with a constant speed of 5 m/s, starting from (–3,4).
   c) Accelerating in the $+x$ direction at 7 m/s^2, starting from (5,3), with an initial velocity of 2 m/s $\measuredangle\, 30°$.
   d) Moving with uniform CCW circular motion, of period 3 s, starting at (–1,0).
   e) Moving with uniform CW circular motion, of period 3 s, starting at (0,–1).

4. The position vector $\mathbf{r}(t)$ of a particle is $\mathbf{r}(t) = 2t\,\mathbf{i} - 5t^2\,\mathbf{j}$. Find the magnitude and direction of the velocity and acceleration at time $t = 2$ s, and describe the motion. Assume MKS units throughout.

5. Identify each of the following motions of a particle of indicated position vector, giving as many quantities of the motion as you can in each case. Also find $\mathbf{v}(t)$ and $\mathbf{a}(t)$ in each case by differentiating, and compare.
   a) $\mathbf{r}(t) = 10\,\mathbf{i}$
   b) $\mathbf{r}(t) = 10t\,\mathbf{i}$
   c) $\mathbf{r}(t) = 4t\,\mathbf{i} + 2t\,\mathbf{j}$
   d) $\mathbf{r}(t) = 3\cos 2\pi t\,\mathbf{i} + 3\sin 2\pi t\,\mathbf{j}$
   e) $\mathbf{r}(t) = 3\sin 2\pi t\,\mathbf{i} + 3\cos 2\pi t\,\mathbf{j}$
   f) $\mathbf{r}(t) = -3\cos 2\pi t\,\mathbf{i} + 3\sin 2\pi t\,\mathbf{j}$

6. Let $\mathbf{A} = 3\,\mathbf{i} + 4\,\mathbf{j}$.
   a) Write a vector **B** that has the same magnitude as **A** but is opposite to **A**.
   b) Write a vector **C** that has twice the magnitude of **A** and is rotated 90° CCW from **A**.
   c) Write a vector $\hat{\mathbf{E}}$ that has the same direction as **A** but has unit magnitude.

7. Show that $\mathbf{A}\cdot(\mathbf{B} + \mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$, that is, that the dot product is distributive.

8. By expanding out $|\mathbf{A} - \mathbf{B}|^2 = (\mathbf{A} - \mathbf{B})\cdot(\mathbf{A} - \mathbf{B})$ and interpreting the result geometrically, prove the law of cosines; namely, $C^2 = A^2 + B^2 - 2AB\cos\theta$.

9. Using the relationship $\cos\theta = \mathbf{A}\cdot\mathbf{B}\,/AB$, find the angle between the two vectors $\mathbf{A} = 3\,\mathbf{i} + 4\,\mathbf{j}$ and $\mathbf{B} = 4\,\mathbf{i} + 3\,\mathbf{j}$. Find a simple way to check your answer.

10. Using two vectors **A** and **B** making angles ϕ_1 and ϕ_2, respectively, with the x axis, prove that $\cos(\phi_2 - \phi_1) = \cos\phi_2\cos\phi_1 + \sin\phi_2\sin\phi_1$.

11. Using two vectors **A** and **B** making angles ϕ and $-\phi$, respectively, with the x axis, prove that $\cos 2\phi = \cos^2\phi - \sin^2\phi$.

12. Find the cross product $\mathbf{C} = \mathbf{A} \times \mathbf{B}$, where $\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and and $\mathbf{B} = \mathbf{i} - \mathbf{j} - \mathbf{k}$. What is the angle between **A** and **B**?

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13. Show that $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$, that is, that the cross product is distributive.

14. Find a unit vector parallel to each of the following vectors:

$\mathbf{A} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{B} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$, and $\mathbf{C} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

15. Find the angle between $\mathbf{A} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{B} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$.

16. Find two vectors perpendicular to both $\mathbf{A} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{B} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$.

17. Show that the following vectors are perpendicular to each other:

$\mathbf{A} = \mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ and $\mathbf{B} = 4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$

18. Re-do the vector products in problems 12, 15, and 17 using the relationship

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

19. Show that the "back cab rule"

$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$ holds for the vectors

$\mathbf{A} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$,

$\mathbf{B} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$, and

$\mathbf{C} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

20. a) Show that the "triple scalar product" $\mathbf{A}\cdot(\mathbf{B} \times \mathbf{C})$ is given by

$$\mathbf{A}\cdot(\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

b) Calculate $\mathbf{A}\cdot(\mathbf{B} \times \mathbf{C})$ for the following vectors:

$\mathbf{A} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$,

$\mathbf{B} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$, and

$\mathbf{C} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

21. Show that the "triple scalar product" $\mathbf{A}\cdot(\mathbf{B} \times \mathbf{C})$ is unchanged by a cyclic interchange of the vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$:

$\mathbf{A}\cdot(\mathbf{B} \times \mathbf{C}) = \mathbf{B}\cdot(\mathbf{C} \times \mathbf{A}) = \mathbf{C}\cdot(\mathbf{A} \times \mathbf{B})$.

Also show that the above result implies that the interchange of the $\cdot$ and $\times$ does not change the value of triple scalar product; namely, $\mathbf{A}\cdot(\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B})\cdot\mathbf{C}$.